so that for a closed system

$$\delta C/\delta\theta = (1/ZRT)\,\delta p/\delta\theta = (1/ZRT)\,\delta P/\delta\theta \quad (6)$$

Combination of Eqs. (1) and (6) results in the following:

$$\delta P/\delta\theta = -K_G ZART(P - P_s)/V \quad (7)$$

which will now be integrated assuming either constant or nonconstant ullage volume.

Case 1. Assume that the ullage volume and the propellant volume are constant, then

$$\delta P/(P - P_s) = (-K_G ZART/V)\,\delta\theta \quad (8)$$

integrating from $P = P_0$ to $P$ and $\theta = 0$ to $\theta$ provides the following result:

$$(P - P_s)/(P_0 - P_s) = \exp(-K_G ZART/V)\theta \quad (9)$$

Case 2. The propellant and ullage volumes are not held constant for this case but are allowed to change in accordance with the ideal gas law for the residual ullage gases

$$V = Z\eta RT/P$$

Assuming that a constant convection rate $\dot{N}$ exists, the residual ullage gas can be determined as $\eta = \eta_0 - \dot{N}\theta$. Substitution for $V$ in Eq. (7) and rearrangement results in the following:

$$\delta P/[P(P - P_s)] = -K_G A\delta\theta/\dot{N}(\eta_0/\dot{N} - \theta) \quad (10)$$

Integrating over the same limits gives

$$[P(P_0 - P_s)]/[P_0(P - P_s)] = (1 - \dot{N}\theta/\eta_0)^{-K_G AP_s/\dot{N}} \quad (11)$$

A comparison of Eqs. (9) and (11) was made by calculating a typical pressure decay with each equation and comparing the results. The results showed a difference of less than 0.4%, indicating that the constant volume assumption is adequate. In addition, it may be deduced from this comparison that the helium absorption process occurs with little change in the ullage and propellant volumes.

White Sands Test Facility (WSTF) data; was used to calculate a mass transfer coefficient ($K_G$) for A-50 and NTO absorption. The calculated $K_G$ values varied from $3.18 \times 10^{-5}$ to $3.35 \times 10^{-5}$ moles/hr m$^2$ atm. Consequently, an average $K_G$ value of $3.27 \times 10^{-5}$ was used for both propellants in conducting this study and was also employed in the resulting computer program.

Use of the technique for describing the helium absorption mechanism shown in this note was made in developing a helium absorption computer program.[2] The use of this program has helped to better understand propellant tank pressure behavior during Apollo missions. Figures 1 and 2 show a comparison of the average Apollo mission tank pressures and the corresponding program predictions. The program has also been used for multiple burn missions.

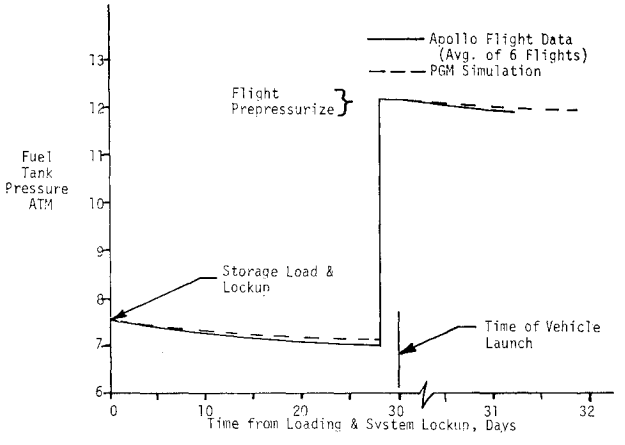


**Fig. 1 Apollo fuel tank pressure history.**

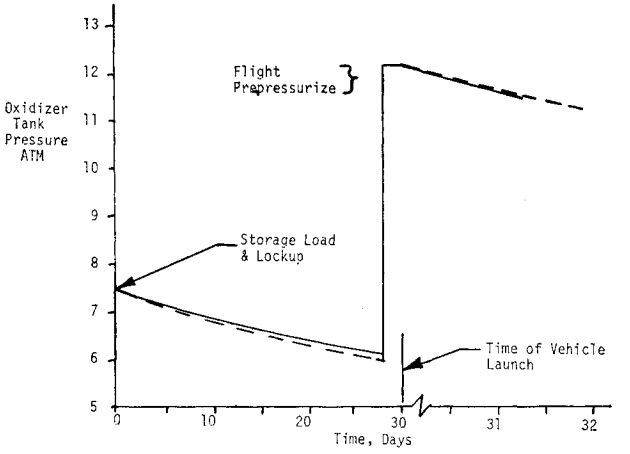


**Fig. 2 Apollo oxidizer tank history.**

### Conclusions

1) Volume changes for the ullage or the propellant are negligible during the helium absorption process. 2) The rate of helium absorption into NTO and A-50 can be adequately simulated as

$$\dot{N}_c = 3{\cdot}27 \cdot 10^{-5}\,(P - P_s)A, \text{ moles/hr}$$

3) The pressure decay of A-50 and/or NTO propellant tanks pressurized with helium can be adequately predicted with Eq. (9)

$$(P - P_s)/(P_0 - P_s) = \exp(-K_G ART/V)\theta \quad (9)$$

### References

[1] Glick, I. J., "Analysis of Simulated KSC Loading and Launch Sequence Performed at White Sands Test Facility, WSTF," Rept. LMO-271-555, Sept. 18, 1969, Grumman Aerospace Corp., Bethpage, N.Y.

[2] Knowles, P. J., "Helium Absorption Computer Program," Rept. 11176-H401-RO-00, Project Apollo, Jan. 8, 1970, TRW Systems, Houston, Texas.

# An Approximation to Midcourse Correction Direction Errors

James F. Kibler* and George R. Young†
*NASA Langley Research Center, Hampton, Va.*

In the statistical analysis of midcourse execution errors, the covariance of $(\bar{v}/v)$, where $\bar{v}$ is the three-dimensional midcourse velocity vector and $v$ is its magnitude, must be determined.[1] This covariance occurs for cases of errors dependent on the direction of the correction velocity vector but independent of its magnitude. The covariance matrix may be determined accurately by numerical integration and in most cases mission analysts resort to this technique. Schmidt[2] gives an exact procedure requiring diagonalization of Cov($\bar{v}$)

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* Mission Applications Section, Mission Analysis and Applications Branch, Space Technology Division.

† Head, Mission Applications Section, Mission Analysis and Applications Branch, Space Technology Division.

for obtaining $\mathrm{Cov}(\bar{v}/v)$, but at the expense of extensive logic and computer time.

For quite some time the approximation[3]

$$\mathrm{Cov}(\bar{v}/v) \approx \mathrm{Cov}(\bar{v})/\mathrm{TRACE}\ \mathrm{Cov}(\bar{v}) \tag{1}$$

has been used for rapid calculation of the covariance. It is the purpose of this note to develop a more accurate approximation to $\mathrm{Cov}(\bar{v}/v)$ that is easily implemented on a computer or readily used by the mission analyst. The new approximation has Eq. (1) as its first-order term and also requires diagonalization of $\mathrm{Cov}(\bar{v})$.

The covariance matrix describing the direction of the velocity correction, $D = \mathrm{Cov}(\bar{v}/v)$, will be approximated assuming that the velocity-to-be gained covariance matrix, $G = \mathrm{Cov}(\bar{v})$, is given, and that the vector $\bar{v}$ has a three-dimensional normal distribution with zero mean and nonzero correlation. A similarity transformation, $\bar{v} = S\bar{\omega}$, may be applied to $G$ to yield a diagonal matrix whose elements are $\lambda_1, \lambda_2, \lambda_3$. $D$ may then be expressed as

$$D = \mathrm{Cov}(\bar{v}/v) = S\,\mathrm{Cov}(\bar{\omega}/\omega)S^T \tag{2}$$

Letting $\bar{\omega} = (xyz)^T$, the problem of finding $D$ is reduced to determining

$$\mathrm{Cov}(\bar{\omega}/\omega) = E(\bar{\omega}\bar{\omega}^T/\omega^2) = \begin{bmatrix} A & O & O \\ O & B & O \\ O & O & C \end{bmatrix} \tag{3}$$

where

$$A = E(x^2/x^2+y^2+z^2),\ B = E(y^2/x^2+y^2+z^2),$$
$$C = E(z^2/x^2+y^2+z^2)$$

and $E(\ )$ is the expected value operator. If an acceptable approximation to $A$ is found, then $B$ and $C$ can be determined by a suitable redefinition of variables.

It is desired to find an approximation to $A$ where $x, y, z$ are independent, normally distributed random variables with zero means and variances $\lambda_1, \lambda_2, \lambda_3$. The exact value of $A$ is

$$A = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{x^2}{x^2+y^2+z^2} f(x, y, z)\,dx\,dy\,dz \tag{4}$$

where the probability density function of the trivariate normal distribution is

$$f(x, y, z) = [(2\pi)^{-3/2}/(\lambda_1\lambda_2\lambda_3)^{1/2}]e^{-1/2(x^2/\lambda_1 + y^2/\lambda_2 + z^2/\lambda_3)} \tag{5}$$

Since analytical integration of Eq. (4) is intractable, a procedure similar to that used by Hoffman and Young[4] is utilized to develop the approximation to $A$. The expression $x^2/(x^2+y^2+z^2)$ is expanded in a Taylor series about a point in a transformed space defined by

$$x^* = |x|,\ y^* = |y|,\ z^* = |z|$$

The expansion point is taken to be $[(a\lambda_1/\pi)^{1/2}, (a\lambda_2/\pi)^{1/2}, (a\lambda_3/\pi)^{1/2}]$ with $a$ to be determined. Truncating after the second-order terms and integrating over the transformed density function yields the approximation

$$A \approx \lambda_1/T + (4/aT^3)[2 - \pi + (2a)^{1/2}]$$
$$[\lambda_1^2(\lambda_2+\lambda_3) - \lambda_1(\lambda_2^2+\lambda_3^2)] \tag{6}$$

where $T = \lambda_1 + \lambda_2 + \lambda_3$ is the trace of $G$.

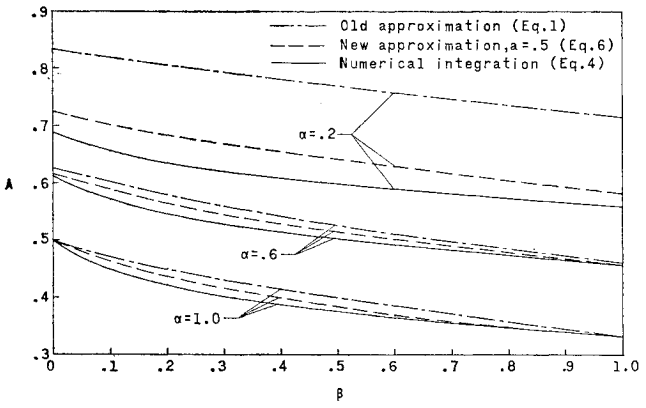


**Fig. 1 Comparison of two approximations and the exact value of $A = E[x^2/(x^2+y^2+z^2)]$ for various eigenvalue ratios.**

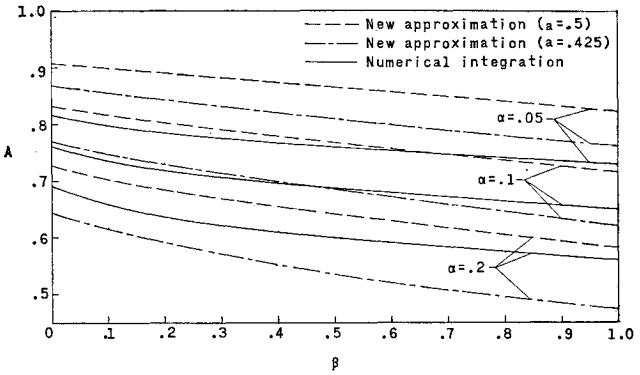


**Fig. 2 Comparison of numerical integration and the new approximation to $A$ with $a = 0.5$ and 0.425.**

Studies indicate that $a = \frac{1}{2}$ yields an excellent approximation to $A$ over most of the range of eigenvalues. In Fig. 1 is presented a comparison of the two approximations [Eqs. (1) and (6)] and the exact value of $A$ [Eq. (4)]. The parameter and abscissa are defined to be $\alpha = \lambda_2/\lambda_1$, $\beta = \lambda_3/\lambda_2$, where the eigenvectors in $S$ have been ordered such that $\lambda_1 < \lambda_2 < \lambda_3$. The "exact" curve was obtained by transforming Eq. (4) to spherical coordinates, integrating the radial component analytically, and then integrating numerically using a very efficient two-dimensional Gaussian integration algorithm with 40 points in each dimension. As a measure of the effort to obtain each curve on a CDC computer, the "exact" curve required about 40 times as much computer time as the new approximation [Eq. (6)]. It can be seen from Fig. 1 that the new approximation consistently yields a better value of $A$ than the approximation obtained from Eq. (1). However, for low values of $\alpha$ the approximation becomes less accurate. Judicious choice of the constant $a$ in Eq. (6) will yield an exact fit for any combination of the eigenvalue ratios $\alpha$ and $\beta$. This, another value of $a$ may be chosen which yields a good approximation to $A$ for small $\alpha$. Studies indicate that $\alpha = 0.425$ yields an acceptable approximation to $A$ for $\alpha < 0.2$. The curves in Fig. 2 illustrate the difference in the new approximation for small $\alpha$. Thus, for two values of the constant $a$, the new approximation to $A$ is closer to the exact value throughout the range of the eigenvalue ratios $\alpha$ and $\beta$. Similar approximations for $B$ and $C$ in Eq. (3) also yield acceptable accuracies.

In conclusion, a new approximation to the components of $\mathrm{Cov}(\bar{v}/v)$ has been developed which is more accurate than a previously used approximation. The effort required to calculate the approximation is much less than that for numerical integration. Two values of the constant $a$ yield an acceptable approximation throughout the range of eigenvalue ratios.

## References

[1] Gates, C. R., "A Simplified Model of Midcourse Maneuver Execution Errors," TR 32-504, Oct. 15, 1963, Jet Propulsion Lab., Pasadena, Calif.

[2] First Quarterly Report by Philco Corp, under Contract No. NAS5-9700 to NASA Goddard Space Flight Center, Greenbelt, Md., Nov. 1964, pp. 5-17–5-40.

[3] White, J. S., Callas, G. P., and Cicolani, L., "Application of Statistical Filter Theory to the Interplanetary Navigation and Guidance Problem," TN D-2697, 1965, NASA, p. 30.

[4] Hoffman, L. H. and Young, G. R., "Approximation to the Statistics of Midcourse Velocity Corrections," TN D-5381, 1969, NASA.